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A New Property of Partitions with Applications to the Rogers-Ramanujan Identities

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ABSTRACT

The order of a partition π (relative to N) is defined as the largest i for which the number of summands in the closed interval $[i, i+N-1]$ is at least i. By studying the generating function for partitions into distinct parts not exceeding $2N$ with given order, we are able to derive an identity of importance in the theory of partitions.

1. INTRODUCTION

The following identity is quite important in the theory of partitions:

$$1 + \sum_{n=1}^{\infty} \frac{(xq;q)_{n-1}(e;q)_n(f;q)_n(1-xq^{2n})(-1)^n(x^2/ef)^n q^{n(3n+1)/2}}{(q;q)_n(xq/e;q)_n(xq/f;q)_n}$$

$$= \Pi\begin{bmatrix} xq, xq/ef; \\ xq/e, xq/f; \end{bmatrix} q \sum_{n=0}^{\infty} \frac{(e;q)_n(f;q)_n}{(q;q)_n}(xq/ef)^n, \tag{1.1}$$

where

$$(x)_n = (x;q)_n = \prod_{j=0}^{n-1}(1-xq^j),$$

$$\Pi\begin{bmatrix} a_1, \ldots, a_r \\ b_1, \ldots, b_s \end{bmatrix}; q = \prod_{i=1}^{r}\prod_{j=0}^{\infty}(1-a_iq^j) \Big/ \prod_{h=1}^{s}\prod_{k=0}^{\infty}(1-b_hq^k).$$

It may be obtained from Watson's q-analog of Whipple's theorem [7, p. 100, eq. (3.4.1.5)] by letting c, d, and $g \to \infty$.

If we let $e \to \infty$ and $f \to \infty$ in (1.1), we obtain the formula from which

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the Rogers-Ramanujan identities are deduced [6, p. 294]. If we replace q by q^2, and then replace e by $-q$, and let $f \to \infty$, we obtain the formula [4, p. 165, eq. (2.16)]from which Göllnitz deduced what are known as the Göllnitz-Gordon identities (cf. [5, p. 741]). If we replace q by q^3, and then set $e = -q$, $f = -q^2$, we obtain a formula [2, eq. (2.16)] from which a partition theorem of Schur may be deduced. Finally by setting $e = \sqrt{q}$, $f = -\sqrt{q}$, it is possible to deduce Theorem 1 of [1].

The object of this paper is to show that a study of partitions into distinct parts utilizing what we shall call the order of a partition allows a relatively short proof of (1.1).

We shall prove the following identity.

$$(-xq; q)_{2N} = \sum_{n=0}^{N} (-xq; q)_{n-1}\, x^n q^{n(3n-1)/2} (1 + xq^{2n}) \begin{bmatrix} N \\ n \end{bmatrix} (-xq^{n+N+1}; q)_{N-n}, \tag{1.2}$$

where

$$\begin{bmatrix} N \\ n \end{bmatrix} = \begin{cases} (q^{N-n+1}; q)_n/(q; q)_n, & 0 \leqslant n \leqslant N, \\ 0, & \text{otherwise.} \end{cases}$$

Previously this identity was deduced by letting b, c, d, and e all $\to \infty$ in the q-analog of Dougall's theorem [7, p. 94, eq. (3.3.1.1)]; equation (i) in [3, p. 5] is equivalent to (1.2).

2. Proof of (1.1)

The coefficient of $x^\mu q^\nu$ on the left side of (1.2) is clearly the number of partitions of ν into μ distinct parts each of which is $\leqslant 2N$.

Let π denote a partition of some number into distinct parts all $\leqslant 2N$. Define $g_i(\pi)$ to be the number of summands of π lying in the closed interval $[i, i + N - 1]$. Clearly $g_{i+1}(\pi) - g_i(\pi) = 0, \pm 1$ for all i. Hence, if $h_i(\pi) = g_i(\pi) - i$, then $h_{i+1}(\pi) - h_i(\pi) = 0, -1, -2$, for all i. Define the order of π to be the largest i for which $h_i(\pi)$ is nonnegative. More simply the order of π is the largest i for which the number of summands in $[i, i + N - 1]$ is $\geqslant i$. Thus, if π is of order n, since $h_0(\pi) \geqslant 0$, and $h_N(\pi) \leqslant 0$, we see that $0 \leqslant n \leqslant N$.

If $n = 0$, then there are clearly no parts of π in $[1, N - 1]$ and indeed N cannot be a summand either or else the maximality of n will be contradicted. Hence the generating function for partitions of order $n = 0$ is

$$(-xq^{n+N+1}; q)_{N-n}.$$

Now, if $n > 0$ is the order of π, then $g_n(\pi) = n$ or $n + 1$ by the maximality of n. Furthermore, if $g_n(\pi) = n + 1$, then n must be a summand of π and $N + n$ must not be a summand of π. Finally, if $g_n(\pi) = n$, then again the maximality of n implies that if $N + n$ is a summand of π, n is also. Thus we have 3 classes of partitions of order $n > 0$ to consider:

Class 1. Order $\pi = n$, $g_n(\pi) = n$, $n + N$ not a summand of π.

Class 2. Order $\pi = n$, $g_n(\pi) = n$, $n + N$ a summand of π.

Class 3. Order $\pi = n$, $g_n(\pi) = n + 1$.

We now derive the generating function for the partions in Class 1. First we recall that

$$x^j q^{j(j+1)/2} \begin{bmatrix} N \\ j \end{bmatrix}$$

is the generating function for partition into j distinct parts each of which is $\leqslant N$ [6, p. 280, Th. 348]. In Class 1, there are exactly n parts in $[n, n + N - 1]$. These are generated by

$$x^n q^{n(n-1)+[n(n+1)/2]} \begin{bmatrix} N \\ n \end{bmatrix}.$$

Since $n + N$ does not appear, the parts $> n + N$ are generated by

$$(-xq^{n+N+1}; q)_{N-n} \,.$$

The parts $< n$ are generated by

$$(-xq; q)_{n-1} \,.$$

Hence the generating function for the partitions in Class 1 is

$$(-xq; q)_{n-1}\, x^n q^{n(3n-1)/2} \begin{bmatrix} N \\ n \end{bmatrix} (-xq^{n+N+1}; q)_{N-n} \,. \tag{2.1}$$

For Class 2 we note that, since $n + N$ is a summand, n must be also. Therefore the parts in the interval $[n, n + N]$ are generated by

$$xq^{n+N} xq^n x^{n-1} q^{(n-1)n+[n(n-1)/2]} \begin{bmatrix} N-1 \\ n-1 \end{bmatrix}.$$

The remaining parts are generated as before and thus the generating function for the partition in Class 2 is

$$q^N(-xq; q)_{n-1}\, x^{n+1} q^{n(3n+1)/2} \begin{bmatrix} N-1 \\ n-1 \end{bmatrix} (-xq^{n+N+1}; q)_{N-n} \,. \tag{2.2}$$

For Class 3 we have noted that n appears and $n + N$ does not appear. Hence the partitions in $[n, n + N]$ are generated by

$$xq^n x^n q^{n^2+[n(n+1)/2]} \begin{bmatrix} N-1 \\ n \end{bmatrix}.$$

Thus we obtain for Class 3 the generating function

$$(-xq; q)_{n-1}\, x^{n+1} q^{n(3n+3)/2} \begin{bmatrix} N-1 \\ n \end{bmatrix} (-xq^{n+N+1}; q)_{N-n}. \tag{2.3}$$

Hence, adding the generating functions of the three classes for all possible orders, we obtain

$$\begin{aligned}(-xq; q)_{2N} &= \sum_{n=0}^{N} (-xq; q)_{n-1}\, x^n q^{n(3n-1)/2} \\ &\quad \times \left(\begin{bmatrix} N \\ n \end{bmatrix} + xq^{2n} \left(q^{N-n} \begin{bmatrix} N-1 \\ n-1 \end{bmatrix} + \begin{bmatrix} N-1 \\ n \end{bmatrix}\right)\right) \\ &\quad \times (-xq^{n+N+1}; q)_{N-n} \\ &= \sum_{n=0}^{N} (-xq; q)_{n-1}\, x^n q^{n(3n-1)/2} \\ &\quad \times \begin{bmatrix} N \\ n \end{bmatrix} (1 + xq^{2n})(-xq^{n+N+1}; q)_{N-n}.\end{aligned}$$

Interpreting $(-xq; q)_{-1}$ as $(1 + x)^{-1}$, we see that the partitions of order $n = 0$ have been treated correctly also. Thus we have (1.2).

3. Derivation of (1.1) from (1.2)

We start with the elegant and elementary Bailey transform [3, p. 1]. If

$$\gamma_n = \sum_{r=n}^{\infty} \delta_r u_{r-n} v_{r+n} \qquad \text{and} \qquad \beta_n = \sum_{r=0}^{n} \alpha_r u_{n-r} v_{n+r},$$

then

$$\sum_{n=0}^{\infty} \alpha_n \gamma_n = \sum_{n=0}^{\infty} \beta_n \delta_n$$

under suitable convergence conditions. For

$$\sum_{n=0}^{\infty} \alpha_n \gamma_n = \sum_{n=0}^{\infty} \sum_{r=n}^{\infty} \alpha_n \delta_r u_{r-n} v_{r+n} = \sum_{r=0}^{\infty} \sum_{n=0}^{r} \alpha_n \delta_r u_{r-n} v_{r+n} = \sum_{r=0}^{\infty} \beta_r \delta_r$$

provided interchange of summation is justified.

Letting $u_n = 1/(q;q)_n$, $v_n = 1/(xq;q)_n$, $\delta_n = (e)_n(f)_n(xq/ef)^n$, and summing the series for γ_n by the q-analog of Gauss's theorem, we obtain (as did Bailey [3, p. 3, eq. (3.1)]),

$$\sum_{n=0}^{\infty} \frac{(e)_n\,(f)_n\,(xq/ef)^n\,\alpha_n}{(xq/e)_n\,(xq/f)_n} = \Pi \begin{bmatrix} xq, xq/ef \\ xq/e, xq/f \end{bmatrix}; q\Big] \sum_{n=0}^{\infty} (e)_n\,(f)_n\,(xq/ef)^n\,\beta_n \tag{3.1}$$

provided

$$\beta_n = \sum_{r=0}^{n} \alpha_n/(xq;q)_{n+r}\,(q;q)_{n-r}\,, \qquad \alpha_0 = 1. \tag{3.2}$$

Let us rewrite (1.2) by dividing both sides by $(-xq;q)_{2N}(q;q)_N$. Thus

$$1/(q;q)_N = \sum_{n=0}^{N} \frac{(-xq;q)_{n-1}\,x^n q^{n(3n-1)/2}\,(1+xq^{2n})}{(q;q)_n\,(q;q)_{N-n}\,(-xq;q)_{N+n}} \tag{3.3}$$

Replacing x by $-x$ in (3.3), we see that we have (3.2) with $\beta_n = 1/(q;q)_n$, and

$$\alpha_n = (-1)^n (xq;q)_{n-1} x^n q^{n(3n-1)/2}(1-xq^{2n})/(q;q)_n\,.$$

These values for α_n and β_n yield the convergence conditions required in Bailey's transform and thus may be substituted into (3.1). This yields (1.1) as indeed Bailey observed in equation (i) of [3, p. 5].

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